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Lagrange Interpolation and Quadrature Formula in Rational Systems

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This paper considers Lagrange interpolation in the rational system $\{1/(x-a_1), 1/(x-a_2), ...\}$, which is based on the zeros of the Chebyshev polynomial for the rational system $\{1, 1/(x-a_1), 1/(x-a_2), ...\}$ with distinct real poles $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}\backslash[-1, 1]$. The corresponding Lebesgue constant is estimated, and is shown to be asymptotically of order $\ln n$ when the poles stay outside an interval which contains $[-1, 1]$ in its interior. Some well-known results of classical polynomial interpolation are extended.

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1. INTRODUCTION

A *Chebyshev system* $\{u_k\}_{k=0}^{n}$ on an interval $[a, b]$ is a set of $n+1$ continuous functions on $[a, b]$ such that any element of $H_n := \text{span}\{u_0, u_1, ..., u_n\}$ that has $n+1$ distinct zeros in $[a, b]$ is identically zero. We say that $\{u_0, u_1, ..., u_n\}$ is a *Markov system* on $[a, b]$ if each $u_i \in C[a, b]$ and $\{u_0, u_1, ..., u_m\}$ is a Chebyshev system for each $m = 0, 1, ..., n$.

For a given Chebyshev system $\{u_k\}_{k=0}^{n}$, we can define the *generalized Chebyshev polynomial* (cf. [4])

$$T_n := \sum_{k=0}^{n} \alpha_k u_k$$

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for H_n on $[a, b]$ by equi-oscillation properties. More precisely, there exists an alternation set of length $n+1$: $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ for T_n on $[a, b]$, that is,

$$T_n(x_k) = (-1)^k \, \|T_n\|_{[a, b]} = (-1)^k, \qquad k = 0, 1, ..., n,$$

where $\|T_n\|_{[a, b]} := \max_{a \leqslant x \leqslant b} |T_n(x)|$ is the uniform norm.

Many extremal problems are solved by the Chebyshev polynomials (cf. [4, 15]) and the denseness of the Markov space is also intimately tied to the location of the zeros of the associated Chebyshev polynomials (cf. [2, Theorem 1]). Chebyshev polynomials are ubiquitous and have many applications (cf. [4, 8, 11, 15]). But, there are very few situations where Chebyshev polynomials can be explicitly computed. However, explicit formulae for the generalized Chebyshev polynomials for the trigonometric rational system

$$\left\{1, \frac{1 \pm \sin t}{\cos t - a_1}, \frac{1 \pm \sin t}{\cos t - a_2}, ..., \frac{1 \pm \sin t}{\cos t - a_n}\right\}, \qquad n = 1, 2, ..., \quad t \in [0, 2\pi), \tag{1.1}$$

and for the rational system

$$\left\{1, \frac{1}{x - a_1}, \frac{1}{x - a_2}, ..., \frac{1}{x - a_n}\right\}, \qquad n = 1, 2, ..., \quad x \in [-1, 1] \tag{1.2}$$

with distinct real poles outside $[-1, 1]$ are implicitly contained in Achiezer's book [1, p. 250]. Recently, Borwein, Erdélyi, and Zhang [5] derived analogue Chebyshev polynomials of the first and second kinds for these systems by constructing certain Blaschke products. It is shown that almost all elementary properties of the classical Chebyshev polynomials hold in this case; for details, one can see [5]. Bultheel, González-Vera, Hendriksen, and Njastad [6] and Djrbashian [7] also considered orthogonal rational functions.

On the other hand, it is not good enough for the approximation of functions to use the classical polynomials in many practical problems. For example, for integrands having poles outside the interval of integration, it would be more natural to design quadrature rules to integrate exactly rational functions (not polynomials), which have the same or almost the same poles, of maximum possible degrees (cf. [9, 18]). Recently, Gautschi [9, 10] has successfully used this idea for the computation of generalized Fermi–Dirac and Bose–Einstein integrals. All of these are closely tied to rational interpolation based on (1.2).

In this paper, we are interested in constructing Lagrange interpolations in the rational system

$$\left\{1, \frac{1}{x-a_1}, \frac{1}{x-a_2}, ..., \frac{1}{x-a_{n-1}}\right\}$$

and

$$\left\{\frac{1}{x-a_1}, \frac{1}{x-a_2}, ..., \frac{1}{x-a_n}\right\}.$$

These interpolations are based on the zeros of Chebyshev polynomials of the first kind for the rational systems (1.2). This extends some well-known results for the classical polynomial interpolation based on the zeros of the classical Chebyshev polynomial of the first kind. More precisely, it is shown that the associated Lebesgue constant is asymptotically of order $\ln n$ when the poles stay outside an interval which contains $[-1, 1]$ in its interior (cf. Theorem 2.1), and this is similar to classical polynomial interpolation. It is also shown that the corresponding mean convergence holds (cf. Theorem 2.3). On the other hand, we obtain a positive quadrature formula that is exact for

$$f \in \operatorname{span}\left\{\frac{1}{x-a_1}, \frac{1}{(x-a_1)^2}, ..., \frac{1}{x-a_n}, \frac{1}{(x-a_n)^2}\right\}$$

(cf. Theorem 2.4). The convergence of this quadrature is also characterized (cf. Theorem 2.5).

This paper is organized as follows. In Section 2 we introduce our notations and formulate the main results. Section 3 contains some auxiliary results which will be used. The proofs of theorems are given in Section 4. The last section gives some remarks.

2. NOTATIONS AND STATEMENTS OF MAIN RESULTS

In this paper, we denote by $\mathbb{P}_n$ the set of all real algebraic polynomials of degree $\leqslant n$. The symbol "$\sim$" is used as follows: if A and B are two expressions depending on some variables and indices, then

$$A \sim B \Leftrightarrow |AB^{-1}| \leqslant c \qquad \text{and} \qquad |A^{-1}B| \leqslant c.$$

Suppose that $\{a_k\}_{k=1}^{n} \subset \mathbb{R}\backslash[-1, 1]$ $(n=1, 2, ...)$ are fixed and distinct. For the rational system

$$\left\{1, \frac{1}{x-a_1}, \frac{1}{x-a_2}, ..., \frac{1}{x-a_n}\right\}, \qquad n=1, 2, ..., \quad x \in [-1, 1], \tag{2.1}$$

we can construct the Chebyshev polynomials as follows (cf. [5]).

We define the numbers $\{c_k\}_{k=1}^{n}$ by

$$a_k := \frac{c_k + c_k^{-1}}{2}, \qquad |c_k| < 1. \tag{2.2}$$

Let

$$M_n(z) := \prod_{k=1}^{n} (z - c_k), \tag{2.3}$$

and

$$f_n(z) := \frac{M_n(z)}{z^n M_n(z^{-1})}. \tag{2.4}$$

Equation (2.4) is actually a finite Blaschke product. The *Chebyshev polynomial of the first kind* for the rational system (2.1) is defined by

$$T_n(x) := \frac{f_n(z) + f_n^{-1}(z)}{2}, \qquad x = \frac{z + z^{-1}}{2}, \qquad |z| = 1, \tag{2.5}$$

while the *Chebyshev polynomial of the second kind* is defined by

$$U_n(x) := \frac{f_n(z) - f_n^{-1}(z)}{z - z^{-1}}, \qquad x = \frac{z + z^{-1}}{2}, \qquad |z| = 1. \tag{2.6}$$

It is shown in [5] that these Chebyshev polynomials preserve almost all the elementary properties of the classical Chebyshev polynomials. Here we just state some of these properties which will be used later.

THEOREM A (cf. [5, Theorem 1.2, Corollary 4.9]). *Assume* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}\backslash[-1, 1]$. *Let* $\{T_n\}_{n=1}^{\infty}$ *and* $\{U_n\}_{n=1}^{\infty}$ *be defined by* (2.5) *and* (2.6), *respectively. Then*

(a) $T_n \in \operatorname{span}\{1, 1/(x-a_1), ..., 1/(x-a_n)\}$.

(b) $\|T_n\|_{[-1,1]} = \|\sqrt{1-x^2}\, U_n(x)\|_{[-1,1]} = 1$.

(c) *$T_n(x)$ has exactly n zeros in $[-1, 1]$:*

$$-1 < x_n < \cdots < x_1 < 1, \tag{2.7}$$

and

$$\sqrt{1-x_k^2}\, U_n(x_k) = \varepsilon(-1)^k, \qquad k = 1, 2, ..., n, \tag{2.8}$$

where $\varepsilon = 1$ or -1.

The conclusion that $T_n(x)$ has exactly n zeros can also be found in [14]. It should be mentioned that $\{T_n(x)\}_{n=1}^{\infty}$ are *not* orthogonal in general (cf. [5, Lemma 4.4]); this property is different from that of the classical Chebyshev polynomials.

In order to state the Bernstein–Markov type inequality, we introduce the function

$$B_n(x) := \sum_{k=1}^{n} \frac{\sqrt{a_k^2 - 1}}{a_k - x}, \tag{2.9}$$

which is called the *Bernstein factor*, where $\sqrt{a_k^2 - 1}$ is defined such that $|c_k| < 1$. The Bernstein factor plays an important role in [5].

THEOREM B (cf. [5, Corollary 3.4, Theorem 3.5]). *Assume the nonreal elements in $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C} \backslash [-1, 1]$ are paired by complex conjugation. Let $p \in \operatorname{span}\{1, 1/(x-a_1), ..., 1/(x-a_n)\}$. Then*

$$|p'(x)| \leqslant \frac{1}{\sqrt{1-x^2}} |B_n(x)|\, \|p\|_{[-1,1]}, \qquad x \in (-1, 1), \tag{2.10}$$

and

$$\|p'\|_{[-1,1]} \leqslant \frac{n}{n-1} \left(\sum_{k=1}^{n} \frac{1+|c_k|}{1-|c_k|} \right)^2 \|p\|_{[-1,1]}, \tag{2.11}$$

and equality holds in (2.10) *if and only if $p(x) = cT_n(x)$, $c \in \mathbb{R}$.*

Let f be a function defined on $[-1, 1]$. We construct the Lagrange interpolation based on the zeros $\{x_k\}_{k=1}^{n}$ of $T_n(x)$ as

$$L_n(f, x) := \sum_{k=1}^{n} f(x_k)\, l_k(x), \tag{2.12}$$

where $\{l_k(x)\}_{k=1}^n$ are the Lagrange fundamental functions

$$l_k(x) := \frac{T_n(x)}{T'_n(x_k)(x-x_k)}, \qquad k=1, ..., n. \tag{2.13}$$

It is easy to check that

$$L_n(f, x_k) = f(x_k), \qquad k=1, ..., n,$$

and $L_n(f, x) \in \operatorname{span}\{1/(x-a_1), ..., 1/(x-a_n)\}$.

We denote by $E_n^R(f)$ the best approximation of $f(x)$ by a linear combination of the functions $\{1/(x-a_1), ..., 1/(x-a_n)\}$ on $[-1, 1]$, that is,

$$E_n^R(f) := \inf_{\beta_k \in \mathbb{R}} \left\| f(x) - \left(\frac{\beta_1}{x-a_1} + \cdots + \frac{\beta_n}{x-a_n} \right) \right\|_{[-1, 1]}. \tag{2.14}$$

It is well known that the Lebesgue constant of classical polynomial interpolation plays an important role in the uniform polynomial approximation. For given $n \in \mathbb{N}$, we also define the associated *Lebesgue function*

$$L_n(x) := \sum_{k=1}^n |l_k(x)|$$

and the *Lebesgue constant*

$$L_n := \max_{x \in [-1, 1]} L_n(x). \tag{2.15}$$

Clearly, it depends on $\{a_k\}_{k=1}^n$.

To simplify the statements of our results, we first introduce an assumption, which plays an important role in the proofs our main results.

DEFINITION 2.0. Let $\{a_k\}_{k=1}^\infty \subset \mathbb{R} \setminus [-1, 1]$ be distinct. If there exists some constant α such that

$$|a_k| \geqslant \alpha > 1, \tag{2.16}$$

i.e., the poles must stay outside an interval which contains $[-1, 1]$ in its interior, we say that $\{a_k\}_{k=1}^\infty \subset \mathbb{R} \setminus [-1, 1]$ satisfy *assumption* (A).

It is easy to see that assumption (A) is equivalent to

$$|c_k| \leqslant \gamma, \qquad k=1, ..., n,$$

where $0 \leqslant \gamma = \alpha - \sqrt{\alpha^2 - 1} < 1$. If this condition is satisfied, we say that $\{c_k\}_{k=1}^\infty$ satisfy *assumption* (C). For convenience, we often use assumption (C) later, instead of assumption (A).

THEOREM 2.1. *Let* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}\backslash[-1, 1]$ *satisfy assumption* (A). *Then*

$$L_n \sim \ln n. \tag{2.17}$$

Remarks. 1. In this case, the corresponding Lebesgue constant for the rational system is asymptotically of order $\ln n$. This is the same as the case of classical polynomial interpolation based on the zeros of the classical Chebyshev polynomial of the first kind (cf. [13, Theorem 2, Vol. III, p. 48], [15], or [16]).

2. Whether (2.17) implies assumption (A) is still open.

With respect to the uniform approximation, we have

COROLLARY 2.2. *Let* $f \in C[-1, 1]$ *and* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}\backslash[-1, 1]$ *satisfy assumption* (A). *Then*

$$\|L_n(f, x) - f(x)\|_{[-1, 1]} \leqslant d_1(\alpha) \ln n E_n^R(f). \tag{2.18}$$

Furthermore, if $f(x)$ *satisfies the Dini–Lipschitz condition*

$$\lim_{\delta \to 0} \omega(f, \delta) \ln \delta = 0,$$

then

$$\lim_{n \to \infty} L_n(f, x) = f(x)$$

uniformly on $[-1, 1]$, *where* $\omega(f, \cdot)$ *is the modulus of continuity of f and, throughout this paper,* $d_i(\alpha)$ $(i = 1, 2, ...)$ *denote some positive constant depending only on* α, *respectively.*

With respect to mean convergence, we have

THEOREM 2.3. *Let* $f \in C[-1, 1]$, $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}\backslash[-1, 1]$, *and* $\{c_k\}$ *be defined by* (2.2). *Then*

$$\|L_n(f, x) - f(x)\|_{v, 2} \leqslant 2\sqrt{\pi}\, E_n^R(f). \tag{2.19}$$

In particular, if $\sum_{k=1}^{\infty} (1 - |c_k|) = \infty$, *then*

$$\|L_n(f, x) - f(x)\|_{v, 2} \to 0, \qquad n \to \infty, \tag{2.20}$$

where

$$\|f\|_{v,2} := \left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |f(x)|^2 \, dx \right)^{1/2}.$$

This extends a result of classical polynomial interpolation (cf. [15, Theorem 1.7, p. 52] or [17]).

Let

$$\lambda_k := \int_{-1}^{1} \frac{l_k(x)}{\sqrt{1-x^2}} \, dx, \qquad k = 1, ..., n. \tag{2.21}$$

We then obtain a *quadrature formula*,

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} \, dx \approx Q_n(f) := \sum_{k=1}^{n} f(x_k) \, \lambda_k. \tag{2.22}$$

We denote its error by

$$E_n^0(f) := \left| \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} \, dx - Q_n(f) \right|. \tag{2.23}$$

With respect to this quadrature formulas we have

THEOREM 2.4. *Let $f \in C[-1, 1]$ and $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1, 1]$. Then*

(a) *$Q_n(f)$ is a positive quadrature formula, that is, $\lambda_k > 0$, $k = 1, ..., n$.*

(b) *For*

$$f \in \operatorname{span} \left\{ \frac{1}{x-a_1}, \frac{1}{(x-a_1)^2}, ..., \frac{1}{x-a_n}, \frac{1}{(x-a_n)^2} \right\},$$

(2.22) *is exact.*

(c) *The error is*

$$E_n^0(f) = O(1) \, E_n^R(f). \tag{2.24}$$

Surprisingly, although our quadrature formula is *neither* Gaussian quadrature *nor* orthogonal quadrature for the rational functions, it is still a positive quadrature formula; moreover, it is *exact* for the $2n$ rational functions

$$\frac{1}{x-a_1}, \frac{1}{(x-a_1)^2}, ..., \frac{1}{x-a_n}, \frac{1}{(x-a_n)^2},$$

while Gaussian and orthogonal quadratures *may not* be exact for the n rational functions

$$\frac{1}{(x-a_1)^2}, \ldots, \frac{1}{(x-a_n)^2}.$$

The Gaussian and orthogonal quadratures based on the rational interpolation were recently considered by Van Assche and Vanherwegen [18] and Gautschi [9].

Theorem 2.5 characterizes the convergence of quadrature formula (2.22).

THEOREM 2.5. *Let* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1, 1]$ *and* $\{c_k\}_{k=1}^{\infty}$ *be defined by* (2.2). *Then*

$$\lim_{n\to\infty} Q_n(f) = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx, \qquad \forall f \in C[-1, 1] \Leftrightarrow \sum_{k=1}^{\infty} (1-|c_k|) = \infty. \tag{2.25}$$

3. AUXILIARY RESULTS

In order to prove the above theorems, we first prove several auxiliary results which will be used later.

LEMMA 3.1. *Let* $f \in C[-1, 1]$, $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1, 1]$, *and* $\{c_k\}$ *be defined by* (2.2). *Then*

$$E_n^R(f) \leqslant E_n(f) + (\|f\|_{[-1,1]} + E_n(f)) \prod_{k=1}^{n} |c_k|, \tag{3.1}$$

where $E_n(f)$ *is the best approximation of* $f(x)$ *by the classical polynomials of degree* $\leqslant n$ *on* $[-1, 1]$.

Proof. Let $p_n(x) := \gamma_n x^n + \cdots + \gamma_0$ $(\gamma_n \neq 0)$ be the best polynomial approximation of degree n to $f(x)$ on $[-1, 1]$, that is,

$$\|p_n(x) - f(x)\|_{[-1,1]} = E_n(f).$$

Using [1, Problem 7, p. 254], we know that there exist μ_k $(k = 1, ..., n)$ such that

$$\left\| \frac{p_n(x)}{\gamma_n} - \frac{1}{\gamma_n} \sum_{k=1}^{n} \frac{\mu_k}{x-a_k} \right\| \leqslant \frac{1}{2^{n-1}} \prod_{k=1}^{n} |c_k|. \tag{3.2}$$

Thus,

$$\left\| f(x) - \sum_{k=1}^{n} \frac{\mu_k}{x - a_k} \right\|_{[-1,1]}$$

$$\leqslant \|f(x) - p_n(x)\|_{[-1,1]} + \left\| p_n(x) - \sum_{k=1}^{n} \frac{\mu_k}{x - a_k} \right\|_{[-1,1]}$$

$$\leqslant E_n(f) + \frac{|\gamma_n|}{2^{n-1}} \prod_{k=1}^{n} |c_k|.$$

Moreover, by Chebyshev inequality (cf. [13, Corollary 2, Vol. I, p. 56]) we have

$$|\gamma_n| \leqslant 2^{n-1} \|p_n\|_{[-1,1]} \leqslant 2^{n-1}(\|f\|_{[-1,1]} + E_n(f)).$$

Then (3.1) follows. ∎

By the classical Jackson theorem (cf. [13, Vol. 1]) and Lemma 3.1 we can prove the following corollary in the usual way.

COROLLARY 3.2. *Let* $f \in C[-1,1]$ *and* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1,1]$ *satisfy assumption* (A). *If* $\prod_{k=1}^{n} |c_k| = O(1/n)$, *then*

$$E_n^R(f) = O(1)(\omega(f, 1/n) + 1/n). \tag{3.3}$$

Lemma 3.3 gives an asymptotic estimate for Bernstein factor (2.9), which plays an important role in the proof of Lemma 3.4.

LEMMA 3.3. *Let* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1,1]$ *satisfy assumption* (A), *and* $B_n(x)$ *be defined by* (2.9). *Then we have*

$$\|B_n\|_{[-1,1]} \sim n. \tag{3.4}$$

Proof. It is easy to check that

$$B_n(x) = \sum_{k=1}^{n} \frac{\sqrt{a_k^2 - 1}}{a_k - x} = \sum_{k=1}^{n} \frac{|a_k - c_k|}{a_k - x}.$$

From the given assumption and by some simple calculations we have

$$\sum_{k=1}^{n} \frac{1 - |c_k|}{1 + |c_k|} \leqslant \sum_{k=1}^{n} \frac{|a_k - c_k|}{|a_k| + 1} \leqslant |B_n(x)| \leqslant \sum_{k=1}^{n} \frac{|a_k - c_k|}{|a_k| - 1} \leqslant \sum_{k=1}^{n} \frac{1 + |c_k|}{1 - |c_k|}. \tag{3.5}$$

Since assumption (A) is equivalent to assumption (C), the conclusion (3.4) follows from (3.5). ∎

In the proof of Lemma 3.5, we need the following lemma.

LEMMA 3.4. *Let* $\{l_k(x)\}_{k=1}^n$ *be defined by* (2.13). *If* $\{a_k\}_{k=1}^\infty \subset \mathbb{R} \setminus [-1, 1]$ *satisfy assumption* (A), *then we have*

$$\sum_{k=1}^n l_k^2(x) \leqslant d_2(\alpha). \tag{3.6}$$

The inequality (3.6) *implies*

$$|l_k(x)| \leqslant \sqrt{d_2(\alpha)}, \qquad k=1, ..., n. \tag{3.7}$$

Proof. Let $\tilde{T}_n(t) := T_n(\cos t)$ and $\tilde{U}_n(t) := U_n(\cos t) \sin t$, where $T_n(x)$ and $U_n(x)$ are the Chebyshev polynomials of the first and second kinds defined by (2.5) and (2.6), respectively. We then have (cf. [5, Theorem 2.1])

$$\tilde{T}'_n(t) = -\tilde{B}_n(t)\, \tilde{U}_n(t), \qquad \tilde{U}'_n(t) = \tilde{B}_n(t)\, \tilde{T}_n(t), \qquad t \in \mathbb{R},$$

where $\tilde{B}_n(t) := B_n(\cos t)$. Hence,

$$T'_n(x) = B_n(x)\, U_n(x), \qquad U'_n(x) = \frac{xU_n(x) - B_n(x)\, T_n(x)}{1-x^2}. \tag{3.8}$$

It follows that

$$T''_n(x) = B'_n(x)\, U_n(x) + B_n(x)\, U'_n(x). \tag{3.9}$$

We may suppose that

$$T_n(x) := \frac{Q_n(x)}{R_n(x)}, \tag{3.10}$$

where $Q_n(x) := e_n(x-x_1) \cdots (x-x_n)$, $R_n(x) := (x-a_1) \cdots (x-a_n)$, and e_n depends on both n and a_k. Then

$$\sum_{k=1}^n \frac{1}{x-x_k} = \frac{T'_n(x)}{T_n(x)} + \frac{R'_n(x)}{R_n(x)} = \frac{T'_n(x)}{T_n(x)} + \sum_{k=1}^n \frac{1}{x-a_k}, \tag{3.11}$$

and

$$\sum_{k=1}^n \frac{1}{(x-x_k)^2} = \frac{(T'_n(x))^2 - T_n(x)\, T''_n(x)}{T_n^2(x)} + \sum_{k=1}^n \frac{1}{(x-a_k)^2}. \tag{3.12}$$

By the given assumption, it is easy to see that

$$\frac{1}{(x-a_k)^2} \leqslant \frac{1}{(|a_k|-1)^2} \leqslant \frac{1}{\alpha^2-1}.$$

Combining (3.8) and Theorem A(c) we have

$$l_k(x) = \frac{T_n(x)}{T'_n(x_k)(x-x_k)} = \varepsilon \frac{(-1)^k \sqrt{1-x_k^2}\, T_n(x)}{B_n(x_k)(x-x_k)},$$

where $\varepsilon = 1$ or -1.

Furthermore, from (3.5) we conclude that

$$\sum_{k=1}^{n} l_k^2(x) \leqslant \left(\frac{1+\gamma}{1-\gamma}\right)^2 \frac{T_n^2(x)}{n^2} \sum_{k=1}^{n} \frac{1-x_k^2}{(x-x_k)^2}.$$

It is easy to check that

$$B'_n(x) = \sum_{k=1}^{n} \frac{\sqrt{a_k^2-1}}{(a_k-x)^2}.$$

We have

$$|B'_n(x)| \leqslant \sum_{k=1}^{n} \frac{\sqrt{a_k^2-1}}{(|a_k|-1)^2} = 2 \sum_{k=1}^{n} \frac{|c_k|}{(1-|c_k|^2)} \leqslant \frac{2\gamma}{1-\gamma^2}\, n. \tag{3.13}$$

Note that

$$\sum_{k=1}^{n} l_k^2(x) \leqslant \left(\frac{1+\gamma}{1-\gamma}\right)^2 \frac{T_n^2(x)}{n^2} \sum_{k=1}^{n} \frac{1-x_k^2}{(x-x_k)^2}.$$

Therefore, combining (3.8)–(3.12), (3.13), and the Markov-type inequality (2.11) and performing some simple calculations, we easily see that

$$\begin{aligned}
\sum_{k=1}^{n} l_k^2(x) &\leqslant \left(\frac{1+\gamma}{1-\gamma}\right)^2 \frac{T_n^2(x)}{n^2} \sum_{k=1}^{n} \frac{1-x^2+x^2-x_k^2}{(x-x_k)^2} \\
&\leqslant \left(\frac{1+\gamma}{1-\gamma}\right)^2 \left\{\frac{T_n^2(x)(1-x^2)}{n^2} \sum_{k=1}^{n} \frac{1}{(x-x_k)^2} + \frac{T_n^2(x)}{n^2}\left(\sum_{k=1}^{n} \frac{2x}{x-x_k} - n\right)\right\} \\
&\leqslant d_2(\alpha),
\end{aligned}$$

and the lemma follows. ∎

Lemma 3.5 gives the estimate of the distance between two consecutive zeros of a Chebyshev polynomial of the first kind, which will be used in the proof of Theorem 2.1.

LEMMA 3.5. (i) *Assume* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1, 1]$. *Let* $\{c_k\}$ *be defined by* (2.2). *If* $\prod_{k=1}^{n} |c_k| = O(1/n)$, *then the largest distance between two consecutive zeros of a Chebyshev polynomial of the first kind satisfies*

$$M_n := \max_{1 \leqslant k \leqslant n+1} |x_k - x_{k+1}| = O(1/n), \tag{3.14}$$

where $x_0 := 1$, $x_{n+1} := -1$. *Moreover, if* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1, 1]$ *satisfy assumption* (A), *then*

$$|x_k - x_{k+1}| \geqslant d_3(\alpha) \frac{1}{n^3}. \tag{3.15}$$

(ii) *Let* $x_k = \cos \theta_k$, *and* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1, 1]$ *satisfy assumption* (A). *Then*

$$|\theta_{k+1} - \theta_k| \sim \frac{1}{n}. \tag{3.16}$$

Proof. First, we prove (3.14). By Corollary 3.2 we know that if $f \in C^1[-1, 1]$, then

$$E_n^R(f) = O(1) \left(\frac{1}{n}\right).$$

Note that any element of span$\{1, 1/(x - a_1), ..., 1/(x - a_n)\}$ has at most n sign changes and $T_n(x)$ has $n + 1$ equi-oscillations. Thus, using [12, Theorem 1, p. 118], it is not difficult to obtain (3.14).

On the other hand, since

$$1 = \left| \frac{l_k(x_k) - l_k(x_{k+1})}{x_k - x_{k+1}} (x_k - x_{k+1}) \right| = |l'_k(\eta)| \, |x_k - x_{k+1}|,$$

(3.15) follows by (2.11) and (3.5).

Now we let $\tilde{l}_k(\theta) := l_k(\cos \theta)$, $k = 1, ..., n$. Note that (cf. the proof of Theorem 2.1) assumption (A) implies that $\prod_{k=1}^{n} |c_k| = O(1/n)$; then, recalling (3.14) and another Markov-type inequality [5, Corollary 3.2], we easily show (3.16) by exactly the same method as that used for the estimates in (3.14) and (3.15). ∎

Lemma 3.6 shows that the corresponding Lagrange fundamental functions $\{l_k(x)\}$ are always orthogonal, which extends a result of the classical polynomial interpolation (cf. [15, Ex. 1.5.20, p. 46]).

LEMMA 3.6. *Let* $\{l_k(x)\}_{k=1}^{n}$ *be defined by* (2.13) *and* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1, 1]$. *If* $k \neq j$, *then*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l_k(x)\, l_j(x)\, dx = 0, \qquad k, j = 1, ..., n. \tag{3.17}$$

Proof. Note that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l_k(x)\, l_j(x)\, dx$$
$$= \frac{1}{T_n'(x_k)\, T_n'(x_j)} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) \frac{T_n(x)}{(x-x_k)(x-x_j)}\, dx.$$

We let

$$\bar{T}_n(x) := \frac{T_n(x)}{(x-x_k)(x-x_j)}.$$

Since $k \neq j$, $\bar{T}_n(x)$ has exactly $n-2$ zeros. This implies that its expansion contains no constant term with respect to rational system (2.1); that is, there exist some constants τ_k, $k = 1, ..., n$, such that

$$\bar{T}_n(x) = \sum_{k=1}^{n} \frac{\tau_k}{x-a_k}.$$

On the other hand, by [5, Corollary 4.6]

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{1}{x-a_k} T_n(x)\, dx = 0, \qquad k = 1, ..., n. \tag{3.18}$$

Thus we can obtain (3.17). ∎

Lemma 3.7 shows that $\{1/(x-a_k)\}_{k=1}^{n}$ are the fixed elements of the generalized Lagrange interpolation $L_n(f, x)$.

LEMMA 3.7. *Assume* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1, 1]$. *Then*

$$L_n\left(\frac{1}{x-a_k}, x\right) \equiv \frac{1}{x-a_k}, \qquad k = 1, ..., n. \tag{3.19}$$

Proof. The proof is straightforward from the defining properties of Chebyshev spaces. However, we here give an elementary proof since we can also prove Lemma 3.8 using the same idea as this. By (3.10) it is easy to check that

$$L_n(f, x) = \frac{1}{R_n(x)} \sum_{k=1}^{n} R_n(x_k)\, f(x_k)\, q_k(x), \tag{3.20}$$

where

$$q_k(x) := \frac{Q_n(x)}{Q_n'(x_k)(x - x_k)}, \qquad k = 1, ..., n. \tag{3.21}$$

Hence, if $f(x) = 1/(x - a_j)$, then $R_n(x)\, f(x)$ is a polynomial of degree $n - 1$. Therefore, by applying the classical Lagrange polynomial interpolation we have

$$\sum_{k=1}^{n} R_n(x_k) \frac{1}{x - a_j} q_k(x) \equiv \frac{R_n(x)}{x - a_j}, \qquad j = 1, ..., n$$

This implies (3.19). ∎

Similar to classical Hermite interpolation, we can also define the *generalized Hermite interpolation* based on the zeros of $T_n(x)$ as

$$H_n(f, x) := \sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} f'(x_k)\, \sigma_k(x), \tag{3.22}$$

where

$$h_k(x) := (1 - 2l_k'(x_k)(x - x_k))\, l_k^2(x), \qquad k = 1, ..., n, \tag{3.23}$$

and

$$\sigma_k(x) := (x - x_k)\, l_k^2(x), \qquad k = 1, ..., n. \tag{3.24}$$

One can verify that

$$H_n(f, x_k) = f(x_k), \qquad k = 1, ..., n,$$

and

$$H_n'(f, x_k) = f'(x_k), \qquad k = 1, ..., n.$$

Here,

$$H_n(f, x) \in \operatorname{span}\left\{\frac{1}{x-a_1}, \frac{1}{(x-a_1)^2}, \ldots, \frac{1}{x-a_n}, \frac{1}{(x-a_n)^2}\right\}.$$

Lemma 3.8 gives the fixed elements of the generalized Hermite interpolation $H_n(f, x)$.

LEMMA 3.8. *Assume* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1, 1]$. *Then*

$$H_n\left(\frac{1}{x-a_k}, x\right) \equiv \frac{1}{x-a_k}, \qquad k=1, \ldots, n, \tag{3.25}$$

and

$$H_n\left(\frac{1}{(x-a_k)^2}, x\right) \equiv \frac{1}{(x-a_k)^2}, \qquad k=1, \ldots, n. \tag{3.26}$$

By the classical polynomial Hermits interpolation and using essentially the same method as that used in Lemma 3.7, it is easy to prove Lemma 3.8. We safely omit it.

LEMMA 3.9. *Assume* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1, 1]$. *Let* $\{c_k\}$ *and* $\{\lambda_k\}$ *be defined by* (2.2) *and* (2.21), *respectively. Then*

$$\lambda_k = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l_k^2(x)\, dx > 0, \qquad k=1, \ldots, n, \tag{3.27}$$

and

$$\sum_{k=1}^{n} \lambda_k = \pi - \frac{2\pi}{1+(c_1 \cdots c_n)^{-2}}. \tag{3.28}$$

Remark. Note that $l_k(x) \in \operatorname{span}\{1/(x-a_1), \ldots, 1/(x-a_n)\}$ $(k=1, \ldots, n)$ imply that $\sum_{k=1}^{n} l_k(x) \not\equiv 1$, which differs from the classical Lagrange polynomial interpolation. Hence, we cannot prove (3.28) in the usual way (cf. [13, Vol. III]). Moreover, (3.28) implies that

$$\sum_{k=1}^{n} \lambda_k < \pi.$$

Proof. Note that $l_k(x)$ has the form

$$l_k(x) = \sum_{k=1}^{n} \frac{\rho_k}{x-a_k}. \tag{3.29}$$

Then, by (3.18) it is easy to show that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, \sigma_k(x)\, dx$$
$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, (x-x_k)\, l_k^2(x)\, dx = 0, \qquad k=1, ..., n.$$

Since Lemma 3.8 yields

$$H_n(l_k(x), x) = l_k(x), \qquad k=1, ..., n,$$

from (3.22)–(3.24) and (3.29) we obtain

$$\lambda_k = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, l_k(x)\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, H_n(l_k(x), x)\, dx$$
$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, l_k^2(x)\, dx > 0, \qquad k=1, ..., n.$$

Equation (3.27) follows. Recall that (cf. [5, Proposition 4.1])

$$T_n(x) = A_0 + \frac{A_1}{x-a_1} + \cdots + \frac{A_n}{x-a_n}, \tag{3.31}$$

where

$$A_0 = \frac{(-1)^n}{2}\, ((c_1 \cdots c_n)^{-1} + c_1 \cdots c_n),$$
$$A_k = \left(\frac{c_k^{-1} - c_k}{2}\right)^2 \prod_{j=1,\, j\neq k}^{n} \frac{1 - c_k c_j}{c_k - c_j}, \qquad k=1, ..., n.$$

Equation (3.31) implies that $T_n(x) - A_0 \in \operatorname{span}\{1/(x-a_1), ..., 1/(x-a_n)\}$. Furthermore, applying Lemma 3.7 we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, (T_n(x) - A_0)\, dx = \sum_{k=1}^{n} \lambda_k (T_n(x_k) - A_0) = -A_0 \sum_{k=1}^{n} \lambda_k. \tag{3.32}$$

Note that (cf. [5, Corollary 4.6, (4.13)])

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, T_n(x)\, dx = (-1)^n\, \pi c_1 \cdots c_n. \tag{3.33}$$

Then, combining (3.32) and (3.33), we can easily show (3.28). ∎

4. PROOFS OF THEOREMS

Proof of Theorem 2.1. First, we claim that assumption (A) implies that $\prod_{k=1}^{n} |c_k| = O(1/n)$. By the inequality

$$\frac{1-x}{1+x} \leqslant e^{-2x}, \qquad x \geqslant 0,$$

and letting $x = (1-|c_k|)/(1+|c_k|)$, we have

$$|c_k| \leqslant \exp\left(-2\,\frac{1-|c_k|}{1+|c_k|}\right).$$

Hence assumption (A) implies that

$$\prod_{k=1}^{n} |c_k| \leqslant \exp\left(-2\sum_{k=1}^{n} \frac{1-|c_k|}{1+|c_k|}\right) \leqslant \exp\left(-2\,\frac{1-\gamma}{1+\gamma}\,n\right).$$

Let $x = \cos\theta$, $x_k = \cos\theta_k$, and $x_j = \cos\theta_j$ be the point nearest to x and let $i = |k-j|$. Then by (3.16) we have

$$d_4(\alpha)\,\frac{i}{n} \leqslant d_5(\alpha)\,|\theta_k-\theta_j| \leqslant |\theta-\theta_k| \leqslant d_6(\alpha)\,|\theta_k-\theta_j| \leqslant d_7(\alpha)\,\frac{i}{n}. \tag{4.1}$$

Since

$$L_n(x) = |T_n(x)| \sum_{k=1}^{n} \frac{\sqrt{1-x_k^2}}{B_n(x_k)\,|x-x_k|},$$

$$\sin\theta_k \leqslant 2\sin\frac{\theta+\theta_k}{2}, \qquad k = 1, ..., n,$$

$$|\cos\theta - \cos\theta_k| = 2\left|\sin\frac{\theta+\theta_k}{2}\,\sin\frac{\theta-\theta_k}{2}\right|,$$

and

$$0 \leqslant \frac{\theta}{\sin\theta/2} \leqslant \pi, \qquad |\theta| \leqslant \pi,$$

by (3.5) one can show that

$$L_n(x) \leqslant |l_j(x)| + \sum_{k \neq j} |l_k(x)| \leqslant \sqrt{d_2(\alpha)} + \frac{1+\alpha}{1-\gamma}\frac{1}{n} \sum_{k \neq j} \frac{\sin \theta_k}{|\cos \theta - \cos \theta_k|}$$

$$\leqslant \sqrt{d_2(\alpha)} + \frac{1+\gamma}{1-\gamma}\frac{1}{n} \sum_{k \neq j} \frac{1}{|\sin(\theta - \theta_k)/2|}$$

$$\leqslant \sqrt{d_2(\alpha)} + \frac{1+\gamma}{1-\gamma}\frac{1}{n}\frac{\pi}{d_4(\alpha)} \sum_{k \neq j} \frac{n}{i},$$

for $x \in [-1, 1]$. Hence

$$L_n = O(\ln n).$$

On the other hand, by (3.5) we have

$$L_n \geqslant L_n(1) \geqslant \frac{1-\gamma}{1+\gamma}\frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{1+x_k}{1-x_k}}.$$

Note that (4.1) implies $\theta_k \sim (k-1)/n$, $k = 1, ..., n$; hence, using a method similar to that used in [15, p. 18], we can prove

$$L_n \geqslant d_8(\alpha) \ln n.$$

Thus, Theorem 2.1 follows. ▮

Proof of Corollary 2.2. Let $p(x)$ be the best approximation for f from span$\{1/(x-a_1), ..., 1/(x-a_n)\}$ on $[-1, 1]$. Then

$$\|p - f\|_{[-1, 1]} \leqslant E_n^R(f). \tag{4.2}$$

Lemma 3.6 yields

$$L_n(f, x) - f(x) = L_n(f - p, x) + (p(x) - f(x)); \tag{4.3}$$

hence, it is easy to obtain (2.18) in the usual way.

Since assumption (A) implies that $\prod_{k=1}^{n} |c_k| = O(1/n)$, Corollary 3.2 implies that $\lim_{n \to \infty} L_n(f, x) = f(x)$ uniformly on $[-1, 1]$. ▮

Proof of Theorem 2.3. By (4.3) and Lemmas 3.6 and 3.9 we have

$$\begin{aligned}
&\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |L_n(f, x)-f(x)|^2 \, dx \\
&\quad \leqslant 2\left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |L_n(f-p, x)|^2 \, dx+\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |p(x)-f(x)|^2 \, dx\right) \\
&\quad \leqslant 2\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \sum_{k=1}^{n} (f(x_k)-p(x_k))^2 \, l_k^2(x) \, dx+2\pi(E_n^R(f))^2 \\
&\quad \leqslant 2(E_n^R(f))^2\left(\sum_{k=1}^{n} \lambda_k+\pi\right) \leqslant 4\pi(E_n^R(f))^2.
\end{aligned}$$

This is (2.19).

On the other hand, $\sum_{k=1}^{\infty} (1-|c_k|)=\infty$ implies that span$\{1/(x-a_1), ..., 1/(x-a_n), ...\}$ is dense on $C[-1, 1]$ (cf. [1, Problem 7, p. 257]); hence we obtain (2.20). ▮

Proof of Theorem 2.4. We may know that Lemma 3.9 implies (a) of Theorem 2.4 and (b) follows from Lemmas 3.7, 3.8, and 3.9 and (3.31). Part (c) is easily obtained. ▮

Proof of Theorem 2.5. Clearly, we may prove the sufficient condition from (2.23). On the other hand, since

$$Q_n(f) \to \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} \, dx, \qquad n \to \infty$$

for every continuous function, by Steklov's theorem (cf. [13, Theorem 4, Vol. III, p. 124]) we know that the systems of interpolating nodes are dense on $[-1, 1]$, that is, $M_n \to 0$ $(n \to \infty)$, where M_n is defined by (3.14). Thus, using Borwein's theorem (cf. [2, Theorem 1]) we have $E_n^R(f) \to 0$ $(n \to \infty)$. Therefore, we may complete the proof of the necessary condition by using [1, Problem 7, p. 254] or [3, Corollary 3]. ▮

5. REMARKS

As we mentioned in Section 3, that $l_k(x) \in \text{span}\{1/(x-a_1), ..., 1/(x-a_n)\}$ implies

$$\sum_{k=1}^{n} l_k(x) \not\equiv 1, \tag{5.1}$$

where $\{l_k(x)\}$ are defined by (2.13).

Therefore, in order to keep the property $\sum_{k=1}^{n} l_k(x) \equiv 1$ we may construct a Lagrange interpolation in another rational system, $\{1, 1/(x-a_1), 1/(x-a_2), ..., 1/(x-a_{n-1})\}$ as follows.

For convenience, we still use $L_n(f, x)$ to denote the Lagrange interpolation operator in the rational system $\{1, 1/(x-a_1), 1/(x-a_2), ...\}$. It is easy to see that

$$L_n(f, x) := \sum_{k=1}^{n} f(x_k) \frac{x-a_n}{x_k-a_n} l_k(x) := \sum_{k=1}^{n} f(x_k) \ell_k(x) \tag{5.2}$$

satisfies

$$L_n(f, x_k) = f(x_k), \qquad k = 1, ..., n,$$

and

$$L_n(f, x) \in \operatorname{span} \left\{1, \frac{1}{x-a_1}, ..., \frac{1}{x-a_{n-1}}\right\},$$

where

$$\ell_k(x) := \frac{x-a_n}{x_k-a_n} l_k(x)$$

is the corresponding Lagrange fundamental function with respect to the rational system $\{1, 1/(x-a_1), ..., 1/(x-a_{n-1})\}$.

One may easily check that if

$$L_n(1, x) \equiv 1, \tag{5.3}$$

and

$$L_n\left(\frac{1}{x-a_i}, x\right) \equiv \frac{1}{x-a_i}, \qquad i = 1, ..., n-1, \tag{5.4}$$

it follows that

$$\sum_{k=1}^{n} \ell_k(x) \equiv 1. \tag{5.5}$$

Note that

$$\left|\frac{x-a_n}{x_k-a_n}\right| \leqslant \frac{|a_n|+1}{|a_n|-1} = \left(\frac{1+|c_k|}{1-|c_k|}\right)^2$$

for $x \in [-1, 1]$. Therefore we have the following approximation theorem with respect to the uniform approximation:

THEOREM 5.1. *Let* $f \in C[-1, 1]$ *and* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1, 1]$ *satisfy assumption* (A). *Then*

$$\|L_n(f, x) - f(x)\|_{[-1,1]} \leqslant d_9(\alpha) \ln n E_{n-1}^{Q}(f). \tag{5.6}$$

Furthermore, if $f(x)$ *satisfies the Dini–Lipschitz condition, then*

$$\lim_{n \to \infty} L_n(f, x) = f(x)$$

uniformly on $[-1, 1]$, *where*

$$E_{n-1}^{Q}(f) := \inf_{\beta_k \in \mathbb{R}} \left\| f(x) - \left(\beta_0 + \frac{\beta_1}{x - a_1} + \cdots + \frac{\beta_{n-1}}{x - a_{n-1}} \right) \right\|_{[-1,1]}. \tag{5.7}$$

The routine of its proof is the same as that of Theorem 2.3; we omit it here.

On the other hand, by some simple calculations we may get

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \ell_k(x) \ell_j(x) \, dx = \frac{\sqrt{1-x_k^2} \sqrt{1-x_j^2}}{B_n(x_k) B_n(x_j)(x_k - a_k)(x_j - a_j)} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n^2(x) \, dx.$$

Hence we conclude that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \ell_k(x) \ell_j(x) \, dx \neq 0. \tag{5.8}$$

Therefore, in order to keep the property $\sum_{k=1}^{n} \ell_k(x) \equiv 1$, we, unfortunately, must destroy such a beautiful orthogonal property as (3.18).

We may also get a similar theorem with respect to mean convergence. We omit the details here.

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